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FINE AND HYPERFINE STRUCTURE OF THE MUONIC ^3He ION

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On the basis of quasipotential approach to the bound state problem in QED we calculate the vacuum polarization, relativistic, recoil, structure corrections of orders α^5 and α^6 to the fine structure interval $\Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2})$ and to the hyperfine structure of the energy levels $2P_{1/2}$ and $2P_{3/2}$ in muonic ^3_2He ion. The resulting values $\Delta E^{fs} = 144803.15\ \ \mu eV$, $\Delta\tilde{E}^{hfs}(2P_{1/2}) = -58712.90\ \ \mu eV$, $\Delta\tilde{E}^{hfs}(2P_{3/2}) = -24290.69\ \ \mu eV$ provide reliable guidelines in performing a comparison with the relevant experimental data.

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I. INTRODUCTION

Simple atoms play important role in the check of quantum electrodynamics (QED), the bound state theory and precise determination of fundamental physical constants (the fine structure constant, the lepton and proton masses, the Rydberg constant, the proton charge radius, etc) [1, 2, 3]. Light muonic atoms (muonic hydrogen (μp), muonic deuterium, ions of muonic helium etc.) are distinguished among simple atoms by the strong influence of the vacuum polarization (VP) effects, recoil effects, nuclear structure and polarizability effects on the structure of the energy levels. The comparison of the theoretical value of the fine and hyperfine splittings in muonic helium ions with the future experimental data will lead to a more precise value of the helion charge radius and the check of quantum electrodynamics with the accuracy 10^{-7}. The energy levels of muonic helium ions were theoretically studied many years ago in [4, 5, 6, 7, 8] both on the basis of the relativistic Dirac equation and nonrelativistic approach, accounting different corrections by the perturbation theory (PT). In these papers the basic contributions to the energies for the $(2P-2S)$ transitions in muonic helium $(\mu^3_2He)^+$ were evaluated with the accuracy 0.1 meV.

In this work we continue the investigation [9] of the energy spectrum of $(\mu^3_2He)^+$ in the P-wave part. The aim of the present study is to calculate such contributions of orders α^5 and α^6 both in the fine and hyperfine structure of the energy states $2P_{1/2}$, $2P_{3/2}$, which are connected with the electron vacuum polarization, the recoil and structure effects, the muon anomalous magnetic moment and the relativistic corrections. The role of all these effects is crucial in order to obtain high theoretical accuracy. Our purpose also consists in the refinement of the earlier performed calculations in [4, 5, 6, 8] and in the derivation of the reliable numerical estimate for the structure of P-wave levels in the ion $(\mu^3_2He)^+$, which can be used for the

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comparison with experimental data. Modern numerical values of fundamental physical constants are taken from Ref.[3]: the electron mass $m_e = 0.510998910(13)\cdot 10^{-3}\ GeV$, the muon mass $m_\mu = 0.1056583668(38)\ GeV$, the fine structure constant $\alpha^{-1} = 137.035999679(94)$, the proton mass $m_p = 0.938272013(23)$ GeV, the helion mass 2.808391383 (70) GeV, the helion magnetic moment $\mu_h = -2.127497723(25)$, the muon anomalous magnetic moment $a_\mu = 1.16592069(60)\cdot 10^{-3}$.

II. FINE STRUCTURE OF P - WAVE ENERGY LEVELS

Our approach to the investigation of the energy spectrum of muonic helium ion $(\mu^3_2He)^+$ is based on the use of quasipotential method in quantum electrodynamics [10, 11, 12], where the two-particle bound state is described by the Schrödinger equation. The basic contribution to the muon and proton interaction operator is determined by the Breit Hamiltonian [13, 14, 15]:

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r} - \frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right) - \tag{1}$$
$$-\frac{Z\alpha}{2m_1m_2r}\left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right) + \Delta V^{fs}(r) + \Delta V^{hfs}(r),$$

where m_1, m_2 are the muon and proton masses, $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass, ΔV^{fs} is the muon spin-orbit interaction:

$$\Delta V^{fs}(r) = \frac{Z\alpha}{4m_1^2r^3}\left[1 + \frac{2m_1}{m_2} + 2a_\mu\left(1 + \frac{m_1}{m_2}\right)\right](\mathbf{L}\boldsymbol{\sigma}_1), \tag{2}$$

ΔV^{hfs} is the helion spin-orbit interaction and the interaction of the muon and helion spins. The leading order $(Z\alpha)^4$ contribution to the fine structure is determined by the operator ΔV^{fs}. As it follows from Eq.(2), the potential ΔV^{fs} includes also the recoil effects (the Barker-Glover correction [16]) and the muon anomalous magnetic moment a_μ correction. The fine structure interval $(2P_{3/2} - 2P_{1/2})$ for the ion $(\mu^3_2He)^+$ can be written in the form:

$$\Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2}) = \frac{\mu^3(Z\alpha)^4}{32m_1^2}\left[1 + \frac{2m_1}{m_2} + 2a_\mu\left(1 + \frac{m_1}{m_2}\right)\right] + \frac{5m_1(Z\alpha)^6}{256} - \tag{3}$$
$$-\frac{m_1^2(Z\alpha)^6}{64m_2} + \frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2}\left[\ln\frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5}\right] + \alpha(Z\alpha)^4A_{VP} + \alpha^2(Z\alpha)^4B_{VP}.$$

This expression includes a relativistic correction of order $(Z\alpha)^6$, which can be calculated with the aid of the Dirac equation [1, 17], the correction of order $\alpha(Z\alpha)^6$ enhanced by the factor $\ln(Z\alpha)$ [18, 19, 20], a number of terms of fifth and sixth order in α which are determined by the effects of the vacuum polarization. The relativistic recoil effects of order $m_1(Z\alpha)^6/m_2$ in the energy spectra of hydrogenic atoms were investigated in Refs.[1, 17, 21, 22, 23]. In the fine splitting (3) they were calculated in [17, 23]. Additional corrections of the same order were obtained in [24]. They do not depend on the muon total momentum j and give the contribution only to the Lamb shift. The contributions to the coefficients A_{VP} and B_{VP} arise in the first and second orders of perturbation theory. Numerical values of the terms in the expression (3), which are presented in the analytical form, are quoted in Table I for definiteness with the accuracy 0.01 μeV. The fine structure interval (3) in

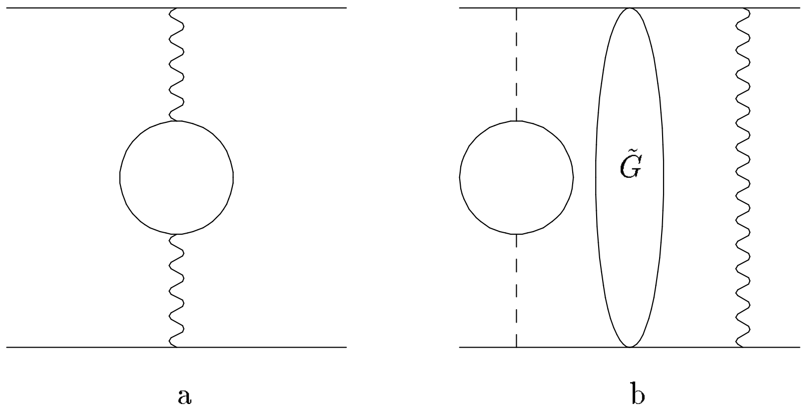

FIG. 1: One-loop vacuum polarization contributions to the fine and hyperfine structure. The dashed line corresponds to the Coulomb interaction. The wave line corresponds to the fine or hyperfine interaction. $\tilde{G}$ is the reduced Coulomb Green's function.

the energy spectrum of electronic hydrogen is considered for a long time as a basic test of quantum electrodynamics [17, 25, 26].

The leading order vacuum polarization potential which gives the contribution to the coefficient A_{VP}, is presented by the Feynman diagrams in Fig.1. The one-loop vacuum polarization effects lead to the modification both the Coulomb interaction and the spin-orbit interaction in expressions (1), (2) [13, 14]:

$$\Delta V^C_{VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(s)ds\left(-\frac{Z\alpha}{r}\right)e^{-2m_esr}, \tag{4}$$

$$\Delta V^{fs}_{VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(s)ds\frac{Z\alpha}{4m_1^2r^3}\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right]e^{-2m_esr}(1+2m_esr)(\mathbf{L}\boldsymbol{\sigma}_1), \tag{5}$$

where the spectral function $\rho(s)=\sqrt{s^2-1}(2s^2+1)/s^4$, m_e is the electron mass. Averaging the potential (2) over the wave functions of the $2P$ - state

$$\psi_{2P}(\mathbf{r}) = \frac{1}{2\sqrt{6}}W^{5/2}re^{-\frac{Wr}{2}}Y_{1m}(\theta,\phi), \quad W=\mu Z\alpha, \tag{6}$$

we obtain the following contribution to the interval (3) (see Fig.1(a)):

$$\Delta E_1^{fs} = \frac{\mu^3(Z\alpha)^4}{32m_1^2}\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right]\times \tag{7}$$

$$\times\frac{\alpha}{3\pi}\int_1^\infty\rho(s)ds\int_0^\infty xdxe^{-x\left(1+\frac{2m_es}{W}\right)}\left(1+\frac{2m_es}{W}x\right) = 129.25 \ \ \mu eV.$$

Although the integral in Eq.(7) can be calculated analytically, we present here for simplicity only its numerical value.

Higher order corrections $\alpha^2(Z\alpha)^4$ entering in the a_μ are taken into account in this expression as well as the recoil effects. The same order contribution $\alpha(Z\alpha)^4$ can be obtained in the second order perturbation theory (see Fig.1(b)). In this case the energy spectrum is determined by the reduced Coulomb Green's function [14, 27, 28]:

$$G_{2P}(\mathbf{r},\mathbf{r}') = -\frac{\mu^2(Z\alpha)}{36z^2z'^2}\left(\frac{3}{4\pi}\mathbf{n}\mathbf{n}'\right)e^{-\frac{z+z'}{2}}g(z,z'), \tag{8}$$

$$g(z,z') = 24z_<^3 + 36z_<^3 z_> + 36z_<^3 z_>^2 + 24z_>^3 + 36z_< z_>^3 + 36z_<^2 z_>^3 + 49z_<^3 z_>^3 - \tag{9}$$
$$-3z_<^4 z_>^3 - 12e_<^z(2 + z_< + z_<^2)z_>^3 - 3z_<^3 z_>^4 + 12z_<^3 z_>^3 \left[-2C + Ei(z_< - \ln(z_<) - \ln(z_>)\right],$$

where $z_< = min(z,z')$, $z_> = max(z,z')$, $C = 0.577216...$ is the Euler constant, $z = Wr$. Using Eqs. (8) and (9) we transform the correction of order $\alpha(Z\alpha)^4$ to the fine structure in the second order perturbation theory as follows:

$$\Delta E_2^{fs} = -\frac{\alpha(Z\alpha)^4\mu^3}{3456\pi m_1 m_2}\left[1 + 2a_\mu + (1 + a_\mu)\frac{2m_1}{m_2}\right] \times \tag{10}$$
$$\times \int_1^\infty \rho(s)ds \int_0^\infty dxe^{-x\left(1+\frac{2m_e s}{W}\right)} \int_0^\infty \frac{dx'}{x'^2}e^{-x'}g(x,x') = 140.56 \quad \mu eV.$$

Let us consider the two-loop vacuum polarization contributions in the one-photon interaction shown in Fig.2. They give the corrections to the fine splitting of P- levels of order $\alpha^2(Z\alpha)^4$.

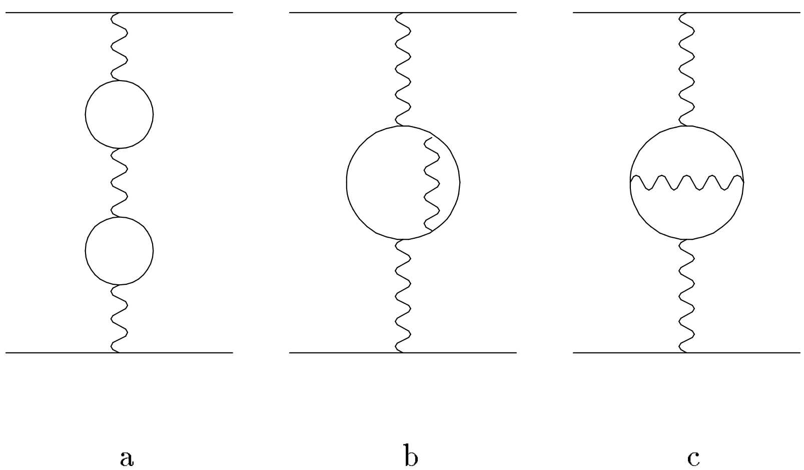

FIG. 2: Effects of two-loop electron vacuum polarization in the one-photon interaction.

In order to obtain the particle-interaction operator for the amplitude, corresponding to the diagram in Fig.2(a), it is necessary to make the substitution

$$\frac{1}{k^2} \to \frac{\alpha}{3\pi}\int_1^\infty ds \frac{\sqrt{s^2-1}(2s^2+1)}{s^4(k^2+4m_e^2 s^2)} \tag{11}$$

two times in the photon propagator. In the coordinate representation, the interaction operator has the form [29, 30]:

$$\Delta V_{VP-VP}^{fs}(r) = \frac{Z\alpha}{r^3}\left[\frac{1+2a_\mu}{4m_1^2} + \frac{1+a_\mu}{2m_1 m_2}\right](\mathbf{L}\boldsymbol{\sigma}_1) \times \tag{12}$$
$$\times \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \frac{1}{(\xi^2-\eta^2)}\left[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r} - \eta^2(1+2m_e\eta r)e^{-2m_e\eta r}\right].$$

Averaging (12) over wave functions (6), we obtain the correction to the interval (3):

$$\Delta E_3^{fs} = \frac{\mu^3\alpha^2(Z\alpha)^4}{72\pi^2}\left[\frac{1+2a_\mu}{4m_1^2} + \frac{1+a_\mu}{2m_1 m_2}\right]\int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \frac{1}{(\xi^2-\eta^2)} \times \tag{13}$$

$$\times \int_0^\infty xdx \left[\xi^2\left(1+\frac{2m_e\xi}{W}x\right)e^{-x\left(1+\frac{2m_e\xi}{W}\right)}-\eta^2\left(1+\frac{2m_e\eta}{W}x\right)e^{-x\left(1+\frac{2m_e\eta}{W}\right)}\right]=0.20\ \ \mu eV.$$

The two-loop vacuum polarization operator is needed to find the interaction operator shown in Fig.2(b,c). The modification of the photon propagator in this case has the form [31]:

$$\frac{1}{k^2}\to\frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{4m_e^2+k^2(1-v^2)}, \tag{14}$$

$$f(v)=v\Biggl\{(3-v^2)(1+v^2)\left[Li_2\left(-\frac{1-v}{1+v}\right)+2Li_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right]+$$

$$\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]+\frac{3}{8}v(5-3v^2)\Biggr\}. \tag{15}$$

The two-loop vacuum polarization potential and the correction to the fine structure $(2P_{3/2}-2P_{1/2})$ are the following:

$$\Delta V^{fs}_{2-loop,VP}(r)=\frac{2\alpha^2(Z\alpha)}{3\pi^2r^3}\left[\frac{1+2a_\mu}{4m_1^2}+\frac{1+a_\mu}{2m_1m_2}\right]\int_0^1\frac{f(v)dv}{1-v^2}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\left(1+\frac{2m_er}{\sqrt{1-v^2}}\right)(\mathbf{L}\boldsymbol{\sigma}_1), \tag{16}$$

$$\Delta E_4^{fs}=\frac{\mu^3\alpha^2(Z\alpha)^4}{12\pi^2}\left[\frac{1+2a_\mu}{4m_1^2}+\frac{1+a_\mu}{2m_1m_2}\right]\times \tag{17}$$

$$\times\int_0^\infty xdx\int_0^1\frac{f(v)dv}{1-v^2}e^{-x\left(1+\frac{2m_e}{W\sqrt{1-v^2}}\right)}\left(1+\frac{2m_e}{W\sqrt{1-v^2}}x\right)=0.78\ \ \mu eV.$$

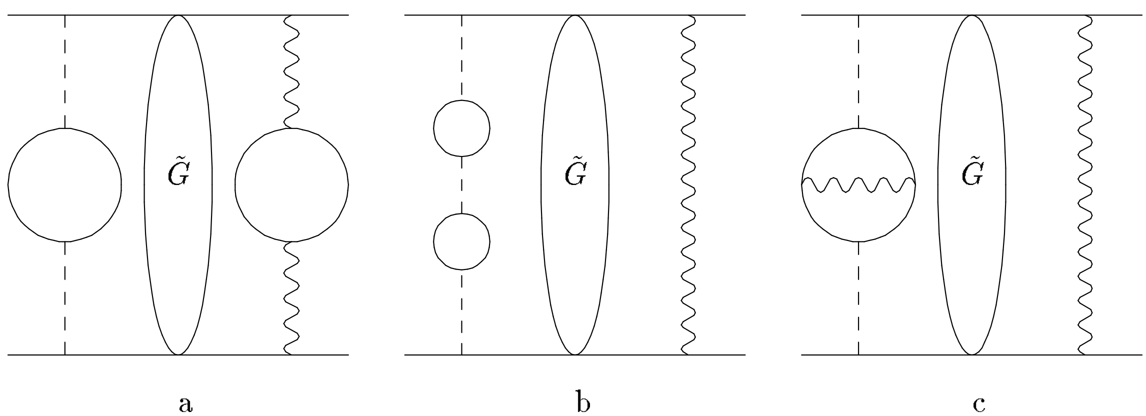

FIG. 3: Effects of two-loop electron vacuum polarization in the second order perturbation theory. The dashed line corresponds to the Coulomb interaction. The wave line corresponds to the fine or hyperfine interaction. $\tilde{G}$ is the reduced Coulomb Green's function.

Two-loop vacuum polarization contributions in the second order perturbation theory shown in Fig.3, have the same order $\alpha^2(Z\alpha)^4$. In order to calculate them, it is necessary to employ relations (2), (4), (5), (8), and the modified Coulomb potential by the two-loop vacuum polarization [10, 11]:

$$\Delta V^C_{VP-VP}(r)=\left(\frac{\alpha}{\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\left(-\frac{Z\alpha}{r}\right)\frac{1}{\xi^2-\eta^2}\left(\xi^2e^{-2m_e\xi r}-\eta^2e^{-2m_e\eta r}\right), \tag{18}$$

$$\Delta V^C_{2-loop,VP}=-\frac{2Z\alpha}{3r}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{1-v^2}e^{-\frac{2m_er}{\sqrt{1-v^2}}}. \quad (19)$$

The amplitude (a) in Fig.3 gives the following correction of order $\alpha^2(Z\alpha)^4$ to the fine splitting:

$$\Delta E_5^{fs}=\frac{\mu^3\alpha^2(Z\alpha)^4}{1296\pi^2}\left[\frac{1+a_\mu}{2m_1m_2}+\frac{1+2a_\mu}{4m_1^2}\right]\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times \quad (20)$$

$$\times\int_0^\infty dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)}\int_0^\infty\frac{dx'}{x'^2}\left(1+\frac{2m_e\eta x'}{W}\right)e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}g(x,x')=0.28\ \ \mu eV.$$

Two other contributions from the amplitudes (b), (c) in Fig.3 have the similar integral structure. Their numerical values are included in Table I. The summary result for the fine splitting ΔE^{fs} in $(\mu_2^3He)^+$ is presented here also. It takes into consideration the numerous earlier performed calculations discussed in the review article [1] and new corrections obtained in this work.

III. HYPERFINE STRUCTURE OF THE ENERGY LEVELS $2P_{1/2}$ AND $2P_{3/2}$

The leading order contribution to the hyperfine splitting of the energy levels $2P_{1/2}$ and $2P_{3/2}$ in muonic helium ion $(\mu_2^3He)^+$ of order α^4 is determined by the potential (the hyperfine part of the Breit potential) [13]:

$$\Delta V_B^{hfs}(r)=\frac{\alpha\mu_h}{2m_1m_2r^3}\left[1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right](\mathbf{L}\boldsymbol{\sigma}_2)- \quad (21)$$

$$-\frac{\alpha\mu_h(1+a_\mu)}{4m_1m_pr^3}\left[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right],$$

where $\mathbf{n}=\mathbf{r}/r$. The operator (21) does not commute with the operator of the muon total angular momentum $\mathbf{J}=\mathbf{L}+\frac{1}{2}\boldsymbol{\sigma}_1$. This leads to the mixing of the $2P_{1/2}$ and $2P_{3/2}$ energy levels and, hence, to a more complicated hyperfine structure of P-wave levels.

In order to calculate the diagonal matrix elements $\langle 2P_{1/2}|\Delta V_B^{hfs}|2P_{1/2}\rangle$ and $\langle 2P_{3/2}|\Delta V_B^{hfs}|2P_{3/2}\rangle$ we can use the following replacements for the operators $(\mathbf{s}_1\mathbf{s}_2)$ and $(\mathbf{L}\mathbf{s}_2)$, which involve the spin of the nucleus [32]:

$$\mathbf{s}_1\to\mathbf{J}\frac{\overline{(\mathbf{s}_1\mathbf{J})}}{J^2},\quad \mathbf{L}\to\mathbf{J}\frac{\overline{(\mathbf{L}\mathbf{J})}}{J^2}, \quad (22)$$

where $\overline{(\mathbf{s}_1\mathbf{J})}$, $\overline{(\mathbf{L}\mathbf{J})}$ are eigenvalues of corresponding operators between the states with equal orbital momentum l. In addition, the averaging over angles in the second term in the right-hand side of (21) can be carried out by means of the relation [32]:

$$\langle\delta_{ij}-3n_in_j\rangle=-\frac{1}{5}(4\delta_{ij}-3\overline{L_iL_j}-3\overline{L_jL_i}). \quad (23)$$

The diagonal matrix elements $\langle 2P_{1/2}|\Delta V^{hfs}|2P_{1/2}\rangle$ and $\langle 2P_{3/2}|\Delta V^{hfs}|2P_{3/2}\rangle$ lead to the following hyperfine structure:

$$\Delta E^{hfs}(2P_{1/2})=E(2^3P_{1/2})-E(2^1P_{1/2})= \quad (24)$$

TABLE I: Fine structure of P-wave energy levels in muonic ${}_2^3He$ ion.

Contribution to the fine splitting ΔE^{fs}	Numerical value of the contribution in μeV	Reference, equation
Contribution of order $(Z\alpha)^4$ $\frac{\mu^3(Z\alpha)^4}{32m_1^2}\left(1+\frac{2m_1}{m_2}\right)$	144186.48	[4, 14], (3)
Muon AMM contribution $\frac{\mu^3(Z\alpha)^4}{16m_1^2}a_\mu\left(1+\frac{m_1}{m_2}\right)$	324.46	[4, 14], (3)
Contribution of order $(Z\alpha)^6$: $\frac{5m_1(Z\alpha)^6}{256}$	19.94	[17, 23], (3)
Contribution of order $(Z\alpha)^6 m_1/m_2$: $-\frac{m_1^2(Z\alpha)^6}{64m_2}$	-0.60	[17, 23], (3)
Contribution of order $\alpha(Z\alpha)^4$ in the first order PT $\langle\Delta V^{fs}_{VP}\rangle$	129.25	[4, 14], (7)
Contribution of one-loop muon VP in the first order PT $\langle\Delta V^{fs}_{MVP}\rangle$	0.01	[4, 14], (7)
Contribution of order $\alpha(Z\alpha)^4$ in the second order PT $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	140.56	(10)
Contribution of order $\alpha(Z\alpha)^6$ $\frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2}\left[\ln\frac{\mu(Z\alpha)^2}{m_1}+\frac{1}{5}\right]$	-0.55	[1, 18, 19]
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{fs}_{VP}\rangle$	0.28	(20)
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{fs}_{VP-VP}\rangle$	0.20	(13)
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{fs}_{2-loop,VP}\rangle$	0.78	(17)
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{VP-VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	0.03	(18)
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{2-loop,VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	2.31	(19)
Summary contribution	144803.15	

$$= E_F\left[\frac{1}{3}+\frac{a_\mu}{6}+\frac{m_1}{6m_2}-\frac{Zm_1m_p}{12m_2^2\mu_h}+\frac{m_1^3}{\mu^3}A^{1/2}_{rel}(Z\alpha)^2+A^{1/2}_{VP}\alpha+B^{1/2}_{VP}\alpha^2\right],$$

$$\Delta E^{hfs}(P_{3/2}) = E(2^5P_{3/2})-E(2^3P_{3/2}) = \tag{25}$$

$$= E_F\left[\frac{2}{15}-\frac{a_\mu}{30}+\frac{m_1}{6m_2}-\frac{Zm_1m_p}{12m_2^2\mu_h}+\frac{m_1^3}{\mu^3}A^{3/2}_{rel}(Z\alpha)^2+A^{3/2}_{VP}\alpha+B^{3/2}_{VP}\alpha^2\right],$$

where $E_F = Z^3\alpha^4\mu^3\mu_h/3m_1m_p$ is the Fermi energy for the $n=2$ level. The calculation of

the relativistic corrections $A_{rel}^{1/2}$, $A_{rel}^{3/2}$ within this approach includes the study of two-photon, three-photon exchange diagrams and the second order perturbation theory contributions with the Breit Hamiltonian determined by Eqs. (1), (2) and (21). More simple method for their calculation is based on the relativistic Dirac equation [20, 33]. In that case, the hyperfine interaction potential has the form:

$$\Delta V_D^{hfs} = e\boldsymbol{\mu}\frac{[\mathbf{r}\times\boldsymbol{\alpha}]}{r^3}. \tag{26}$$

Its contributions to the hyperfine splitting are given by

$$\Delta E_{rel}^{hfs}(2P_{1/2}) = \frac{4\alpha\mu_h}{m_p}R_{1/2}, \tag{27}$$

$$\Delta E_{rel}^{hfs}(2P_{3/2}) = -\frac{16\alpha\mu_h}{15m_p}R_{3/2},$$

where the nuclear magnetic moment $\boldsymbol{\mu} = g_N\mu_N\mathbf{s}_2$ ($\mu_N = e/2m_p$). The radial integrals $R_k = \int_0^\infty g_k f_k dr$, which are characteristic of this case, are determined by the Dirac wave functions f_k, g_k of the states $2P_{1/2}$ and $2P_{3/2}$. Taking into account their explicit form [32], we obtain the following values of the relativistic corrections to the hyperfine structure of the P - wave levels:

$$A_{rel}^{1/2} = \frac{47}{72}, \quad A_{rel}^{3/2} = \frac{7}{180}. \tag{28}$$

These values of the coefficients coincide with the result obtained in Ref.[23] by analytically calculating the contribution of order $m_1^2(Z\alpha)^6/m_2$ to the hyperfine structure of the P - wave levels of the hydrogen atom at $n = 2$.

The fifth order contribution over α due to the electron vacuum polarization (see diagrams (a), (b) in Fig.1) appears in the hyperfine splitting in just the same way as in the fine structure of the spectrum. The modification of the hyperfine part of the Breit potential induced by the vacuum polarization is described by the following expression (the substitution (11) is used) [14]:

$$\Delta V_{VP}^{hfs}(r) = \frac{\alpha\mu_h}{2m_1m_pr^3}\left[1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right](\mathbf{L}\boldsymbol{\sigma}_2)\int_1^\infty\rho(s)dse^{-2m_esr}\left(1+2m_esr\right)- \tag{29}$$

$$-\frac{\alpha\mu_h(1+a_\mu)}{4m_1m_pr^3}\int_1^\infty\rho(s)dse^{-2m_esr}\Big[4m_e^2s^2r^2\left(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right)+$$

$$+(1+2m_esr)\left(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right)\Big].$$

The subsequent transformations of the diagonal matrix elements of the operator (29), connected with the averaging over angles, can be performed with the use of the formula (23) for the second term in the square brackets of Eq.(29). Similar averaging for the first term in the square brackets is:

$$\langle\delta_{ij}-n_in_j\rangle = \frac{1}{5}(2\delta_{ij}+\overline{L_iL_j}+\overline{L_jL_i}). \tag{30}$$

Then the contributions of the vacuum polarization in the first and second orders of perturbation theory can be written as follows:

$$\Delta E_1^{hfs}(2P_{1/2}) = E_F\frac{\alpha}{18\pi}\int_1^\infty\rho(s)ds\int_0^\infty xdxe^{-x\left(1+\frac{2m_es}{W}\right)}\times \tag{31}$$

$$\times\left[\left(1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right)\left(1+\frac{2m_es}{W}x\right)+(1+a_\mu)\left(\frac{2m_e^2s^2x^2}{W^2}+1+\frac{2m_esx}{W}\right)\right]=-65.21\ \ \mu eV,$$

$$\Delta E_1^{hfs}(2P_{3/2})=E_F\frac{\alpha}{18\pi}\int_1^\infty\rho(s)ds\int_0^\infty xdxe^{-x\left(1+\frac{2m_es}{W}\right)}\times \tag{32}$$

$$\times\left[\left(1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2}\right)\left(1+\frac{2m_es}{W}x\right)-\frac{(1+a_\mu)}{5}\left(\frac{8m_e^2s^2x^2}{W^2}+1+\frac{2m_esx}{W}\right)\right]=-11.15\ \ \mu eV,$$

$$\Delta E_2^{hfs}(2P_{1/2})=E_F\frac{\alpha}{324\pi}\int_1^\infty\rho(s)ds\int_0^\infty dxe^{-x\left(1+\frac{2m_es}{W}\right)}\times \tag{33}$$

$$\times\int_0^\infty\frac{dx'}{x'^2}e^{-x'}g(x,x')\left[2+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}+a_\mu\right]=-56.79\ \ \mu eV,$$

$$\Delta E_2^{hfs}(2P_{3/2})=E_F\frac{\alpha}{324\pi}\int_1^\infty\rho(s)ds\int_0^\infty dxe^{-x\left(1+\frac{2m_es}{W}\right)}\times \tag{34}$$

$$\times\int_0^\infty\frac{dx'}{x'^2}e^{-x'}g(x,x')\left[\frac{4}{5}+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}-\frac{a_\mu}{5}\right]=-23.42\ \ \mu eV.$$

Two-loop vacuum polarization corrections to the hyperfine part of the potential for the $l\neq 0$ can be obtained with the aid of relations (11) and (14). The results are

$$\Delta V_{VP-VP}^{hfs}(r)=\frac{Z\alpha\mu_h}{2m_1m_2r^3}\left(\frac{\alpha}{\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\frac{1}{\xi^2-\eta^2}\times \tag{35}$$

$$\times\Biggl\{\left[1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right](\mathbf{L}\boldsymbol{\sigma}_2)\left[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r}-\eta^2(1+2m_e\eta r)e^{-2m_e\eta r}\right]-$$

$$-\frac{1+a_\mu}{2}\Bigl[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n}))\left(\xi^2(1+2m_e\xi r)e^{-2m_e\xi r}-\eta^2(1+2m_e\eta r)e^{-2m_e\eta r}\right)+$$

$$+4m_e^2r^2\left(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right)\left(\xi^4e^{-2m_e\xi r}-\eta^4e^{-2m_e\eta r}\right)\Bigr]\Biggr\},$$

$$\Delta V_{2-loop,VP}^{hfs}(r)=\frac{Z\alpha\mu_h}{2m_1m_2r^3}\frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{1-v^2}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\times \tag{36}$$

$$\times\Biggl\{\left[1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right]\left(1+\frac{2m_er}{\sqrt{1-v^2}}\right)(\mathbf{L}\boldsymbol{\sigma}_2)-\frac{1+a_\mu}{2}\times$$

$$\times\left[\frac{4m_e^2r^2}{1-v^2}\left(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right)+\left(1+\frac{2m_er}{\sqrt{1-v^2}}\right)\left(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right)\right]\Biggr\}.$$

Omitting further details of the calculations that can be performed with the aid of a procedure similar to that which was used in the derivation (17) and (20), we represent in Table II the numerical values of the contributions to the energy spectrum that are determined by the potentials (35) and (36). Yet another part of two-loop corrections to the hyperfine structure in the second order PT is shown in Fig.3. We also included in Table II numerical results of the contributions from these amplitudes to hyperfine structure of P-wave states.

Nuclear structure effects play significant role in the precise calculation of the hyperfine structure in muonic atoms [9, 10]. In order to calculate it in the case of P-wave levels we can expand the helion magnetic form factor over relative momenta and obtain the hyperfine

TABLE II: Hyperfine structure of P-wave energy levels in muonic helium ion $(\mu^3_2He)^+$.

Contribution to hyperfine splitting	Numerical value of the contribution to $\Delta E^{hfs}(2P_{1/2})$, μeV	Numerical value of the contribution to $\Delta E^{hfs}(2P_{3/2})$, μeV	Reference, equation
Contribution of order α^4	-58356.61	-24088.50	[4, 14], (24), (25)
Muon AMM contribution	-33.29	6.66	[4], (24), (25)
Relativistic correction of order α^6	-26.62	-1.59	(28)
Contribution of order α^5 in the first order PT $\langle\Delta V^{hfs}_{VP}\rangle$	-65.21	-11.15	[4], (31),(32)
Contribution of order α^5 in the second order PT $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{hfs}\rangle$	-56.79	-23.42	(33),(34)
VP Contribution in the second order PT of order α^6 $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{hfs}_{VP}\rangle$	-0.12	-0.07	(4),(29)
VP Contribution of 1γ interaction of order α^6 $\langle\Delta V^{hfs}_{VP-VP}\rangle$	-0.10	-0.01	(35)
VP Contribution of 1γ interaction of order α^6 $\langle\Delta V^{hfs}_{2-loop,VP}\rangle$	-0.37	-0.08	(36)
VP Contribution in the second order PT of order α^6 $\langle\Delta V^C_{VP-VP}\cdot\tilde G\cdot\Delta V^{hfs}\rangle$	-0.01	-0.004	(11),(18),(21)
VP Contribution in the second order PT of order α^6 $\langle\Delta V^C_{2-loop,VP}\cdot\tilde G\cdot\Delta V^{hfs}\rangle$	-0.45	-0.19	(11),(19),(21)
Nuclear structure correction	-0.33	0.66	(38)
Summary contribution	-58539.90	-24117.69	

part of the interaction operator in the momentum representation, which is proportional to magnetic mean-square radius $< r_M^2 >$ of the helion:

$$\Delta V^{hfs}_{str}(\mathbf{k}) = -\frac{\pi Z\alpha(1+a_\mu) < r_M^2 >}{6m_1m_2}\left[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\mathbf{k}^2-(\boldsymbol{\sigma}_1\mathbf{k})(\boldsymbol{\sigma}_2\mathbf{k})\right], \quad (37)$$

Averaging this operator over the wave functions ψ_{2P} we obtain the following contribution to the hyperfine structure:

$$\Delta E^{hfs}_{str} = \frac{\mu^5(Z\alpha)^6(1+a_\mu) < r_M^2 >}{144m_1m_2}\frac{[F(F+1)-j(j+1)-\frac{3}{4}][j(j+1)-\frac{5}{4}]}{j(j+1)}. \quad (38)$$

Corresponding numerical values for the hyperfine splittings of $2P_{1/2}$ and $2P_{3/2}$ states are included in Table II. We assume that the values of the charge and magnetic mean-square radii coincide and take the value of ^3_2He charge radius $r_E = 1.9642(11)$ fm [34].

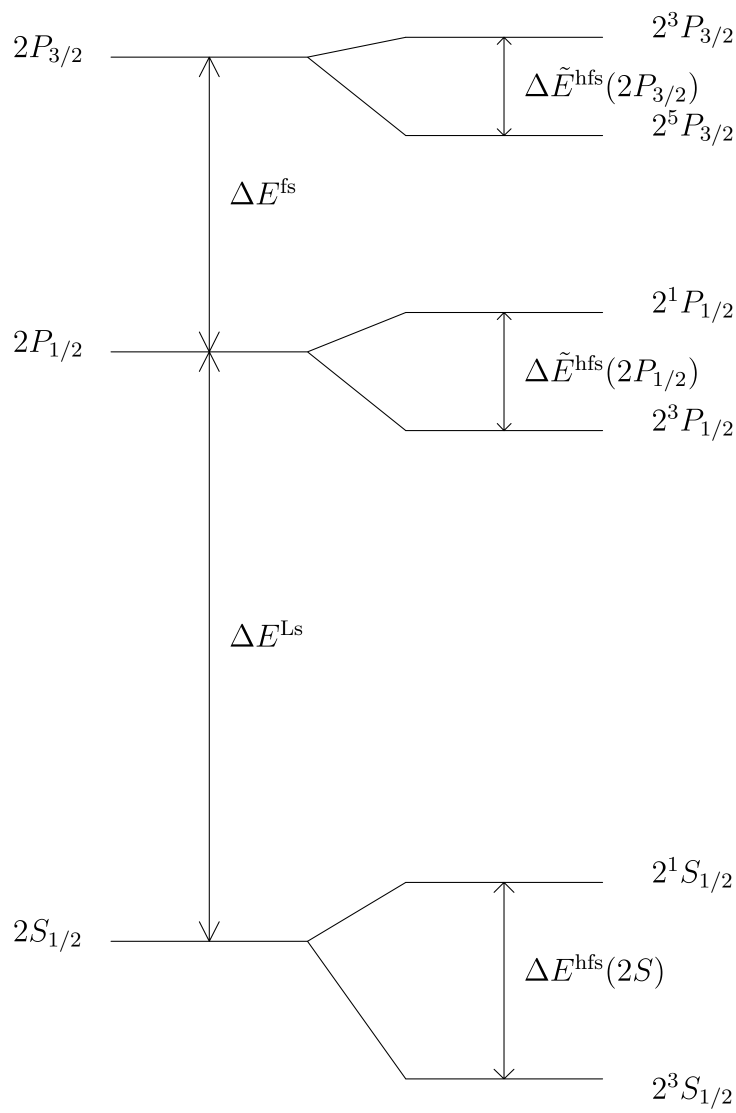

FIG. 4: The structure of S-wave and P-wave energy levels in muonic helium ion $(\mu_2^3He)^+$ for the $n = 2$.

The off-diagonal matrix element has an important role to attain the high accuracy of the calculation of the P-wave levels in muonic helium ion. We present its general structure as follows:

$$\gamma = \langle 2^3P_{1/2}|\Delta V^{hfs}|2^3P_{3/2}\rangle = E_F\left(-\frac{\sqrt{2}}{48}\right)\left[1-a_\mu+\frac{2m_1}{m_2}-\frac{Zm_1m_p}{m_2^2\mu_h}+\frac{m_1^3}{\mu^3}C_{rel}(Z\alpha)^2+C_{VP}\alpha\right], \tag{39}$$

where for the sake of simplicity, we have restricted ourselves to terms of fifth order in α in considering vacuum polarization effects, terms of fifth and higher orders in the muon anomalous magnetic moment, relativistic effects of order $(Z\alpha)^6$ and recoil effects. The first three terms in the right-hand side of Eq. (37) result from employing the potential (21). In the Dirac's theory, relativistic corrections are determined by off-diagonal radial integrals:

$$R_{\frac{1}{2}\frac{3}{2}} = \int_0^\infty \left(g_{\frac{1}{2}}(r)f_{\frac{3}{2}}(r)+g_{\frac{3}{2}}(r)f_{\frac{1}{2}}(r)\right)dr. \tag{40}$$

By using the explicit expressions for the wave functions $f_{1/2,3/2}(r)$ and $g_{1/2,3/2}(r)$ [32] in order to calculate this integral, we obtain the coefficient $C_{rel} = 9/16$. In order to obtain the vacuum polarization correction in γ we use the potential (29). Then we have to calculate the matrix elements of the following operators:

$$T_1 = (\mathbf{L}\boldsymbol{\sigma}_2),\ T_2 = \left[\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - 3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right],\ T_3 = \left[\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - (\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})\right]. \tag{41}$$

Upon averaging over angles with the aid of expressions (23) and (30), they can be written in terms of the $6j$ - symbols as

$$\langle T_3\rangle = -\langle T_2\rangle = -\langle T_1\rangle == 6\hat{j}\hat{j}'\begin{Bmatrix} l & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j \end{Bmatrix}\begin{Bmatrix} l & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j' \end{Bmatrix} = \frac{2\sqrt{2}}{3}, \tag{42}$$

where the value of the total momentum $F = 1$ ($\mathbf{F} = \mathbf{s}_2 + \mathbf{J}$), $l = 1$, $\hat{j} = \sqrt{2j+1}$, $\hat{j}' = \sqrt{2j'+1}$, and the numerical values of the $6j$ - symbols are borrowed from [35]. As a result the vacuum polarization contributions to the off-diagonal matrix element (39) in the first and second orders of perturbation theory have the form:

$$\gamma_1 = \langle 2^3P_{1/2}|\Delta V^{hfs}_{VP}|2^3P_{3/2}\rangle = E_F\left(-\frac{\sqrt{2}}{72}\right)\frac{\alpha}{\pi}\int_1^\infty \rho(s)ds\int_0^\infty xdxe^{-x\left(1+\frac{2m_es}{W}\right)}\times \tag{43}$$

$$\times\left[\left(1+\frac{m_1}{m_2}-\frac{Zm_1m_p}{2m_2^2\mu_h}\right)\left(1+\frac{2m_es}{W}x\right)-\frac{1+a_\mu}{2}\left(1+\frac{2m_es}{W}x-\frac{4m_e^2s^2}{W^2}x^2\right)\right] = 9.49\ \mu eV,$$

$$\gamma_2 = \langle 2^3P_{1/2}|\Delta V^C_{VP}\cdot\tilde{G}\cdot\Delta V^{hfs}_B|2^3P_{3/2}\rangle = E_F\left(-\frac{\sqrt{2}}{2592}\right)\frac{\alpha}{\pi}\left[1+\frac{2m_1}{m_2}-\frac{Zm_1m_p}{m_2^2\mu_h}-a_\mu\right]\times \tag{44}$$

$$\times\int_1^\infty \rho(s)ds\int_0^\infty xdxe^{-x\left(1+\frac{2m_es}{W}\right)}\int_0^\infty\frac{dx'}{x'^2}e^{-x'}g(x,x') = 5.33\ \mu eV.$$

The total numerical value of the matrix element (39) is $\gamma = 5497.28\ \mu eV$. It leads to the shift of the hyperfine splittings of the energy levels $2^3P_{3/2}$ and $2^3P_{1/2}$ by the value $\delta = 173.0\ \mu eV$.

IV. SUMMARY AND CONCLUSION

In the present study we have calculated QED effects in the fine and hyperfine structure of the $2P_{1/2}$ and $2P_{3/2}$ energy levels in muonic helium ion $(\mu_2^3He)^+$. We have considered the electron vacuum polarization contributions of orders α^5, α^6, recoil corrections, relativistic effects of order α^6 and nuclear structure corrections. The numerical values of the contributions are presented in Tables I and II. In these Tables we give the references to other papers also devoted to the investigation of the structure of P-wave levels in the hydrogenic atoms.

Let us summarize the basic particularities of the calculation performed above.

1. Special attention in our investigation has been concentrated on the vacuum polarization effects. For this purpose we obtain the terms of the interaction operator in muonic helium ion which contain the one-loop and two-loop vacuum polarization corrections.

2. In each order in α we have taken into account recoil effects in the terms proportional to m_1/m_2. The experimental values of the muon and helion magnetic moments are used [3].

3. The calculation of the relativistic corrections to the diagonal and nondiagonal matrix elements both for the fine and hyperfine structure intervals is performed on the basis of the Dirac equation.

TABLE III: $(2S-2P)$ transition energies for muonic helium ion $(\mu_2^3He)^+$.

Transition	Energy (meV)	[5]
${}^1S_{1/2} - {}^3P_{1/2}$	1167.33	1167.3
${}^1S_{1/2} - {}^3P_{3/2}$	1342.19	1342.3
${}^3S_{1/2} - {}^1P_{1/2}$	1392.66	1392.4
${}^3S_{1/2} - {}^3P_{1/2}$	1333.95	1334.1
${}^3S_{1/2} - {}^3P_{3/2}$	1508.80	1509.1
${}^3S_{1/2} - {}^5P_{3/2}$	1484.51	1484.6

Total numerical values for the fine structure interval ΔE^{fs} (3) and hyperfine splitting intervals of $2P_{1/2}$ and $2P_{3/2}$ states are presented in Tables I,II. Taking also into account our calculation of the mixing of the $2^3P_{3/2}-$ and $2^3P_{1/2}-$ wave energy levels (the correction δ), we find that these values of hyperfine splittings change by $\delta = 173.0\ \mu eV$: $\Delta\tilde{E}^{hfs}(2P_{1/2}) = \Delta E^{hfs}(2P_{1/2}) - \delta = -58712.90\ \mu eV$, $\Delta\tilde{E}^{hfs}(2P_{3/2}) = \Delta E^{hfs}(2P_{3/2}) - \delta = -24290.69\ \mu eV$. The theoretical error of the obtained results is determined by the contributions of higher order and amounts up to 10^{-6}. Previously, the $(2S-2P)$ transition energies for muonic helium ion were studied in [4, 5, 6]. Considering the obtained in [5, 6] numerical results for different transitions $(2s-2p)$ we find that the fine splitting and the hyperfine splitting intervals for the states $2P_{1/2}$ and $2P_{3/2}$ in this paper are equal: $\Delta E^{fs} = 145.0\ meV$, $\Delta E^{hfs}(2P_{1/2}) = 58.3\ meV$, $\Delta E^{hfs}(2P_{3/2}) = 24.5\ meV$. So, the results of our work agree and refine the previous calculations performed in [5, 6] via taking into account higher order effects. They can be considered as a reliable estimate for the fine and hyperfine structure intervals for the P- levels in muonic helium ion $(\mu_2^3He)^+$. The disposition of the P-wave energy levels is shown in Fig.4. Taking into account the value of the Lamb shift $(2S-2P)$ from [4], our result for the hyperfine splitting of $2S$ state from [9] and the numerical results obtained in this work, we find new $(2S-2P)$ transition energies, which are presented in Table III.

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[1] M.I.Eides, H.Grotch, V.A.Shelyuto, Phys. Rep. **342**, 62 (2001).
[2] S.G.Karshenboim, Phys. Rep. **422**, 1 (2005).
[3] P.J.Mohr, B.N.Taylor, D.B. Newell, Rev. Mod. Phys. **80**, 633 (2008).
[4] E.Borie, G.A.Rinker, Rev. Mod. Phys. **54**, 67 (1982).
[5] E.Borie, Z. Phys. A **278**, 127 (1976).
[6] E.Borie, Z. Phys. A **297**, 17 (1980).
[7] S.V.Romanov, Z.Phys. D **28**, 7 (1993).
[8] R.Swainson, G.W.F.Drake, Phys. Rev. A **34**, 620 (1986).

[9] A.P.Martynenko, JETP **106**, 691 (2008).
[10] R.N.Faustov, A.P.Martynenko, JETP **98**, 39 (2004).
[11] R.N.Faustov, A.P.Martynenko, Phys. Rev. **A67**, 052506 (2003).
[12] A.P. Martynenko, JETP **101**, 1021 (2005).
[13] V.B. Berestetskii, E.M. Lifshits, L.P. Pitaevskii, *Quantum Electrodynamics*, Moscow, Nauka, 1980.
[14] K. Pachucki, Phys. Rev. **A53**, 2092 (1996).
[15] A. Veitia, K. Pachucki, Phys. Rev. A **69**, 042501 (2004).
[16] W.A. Barker, F.N. Glover, Phys. Rev. **99**, 317 (1955).
[17] J.R. Sapirstein, D.R. Yennie, in *Quantum Electrodynamics*, edited by T.Kinoshita, World Scientific, Singapore, p. 560, (1990).
[18] G.W. Erickson, D.R. Yennie, Ann. Phys. **35**, 271 (1965).
[19] N.L. Manakov, A.A. Nekipelov, A.G. Fainshtein, JETP **95**, 1167 (1989).
[20] G.W. Erickson, D.R. Yennie, Ann. Phys. **35**, 447 (1965).
[21] V.M. Shabaev, A.N. Artemyev, T. Beier, G. Soff, J. Phys. B **31**, L337 (1998).
[22] A.N. Artemyev, V.M. Shabaev, V.A. Yerokhin, Phys. Rev. A **52**, 1884 (1995).
[23] E.A. Golosov, A.S. Yelkhovsky, A.I. Milshtein, I.B.Khriplovich, ZhETF **107**, 393 (1995).
[24] U.Jentschura, K.Pachucki, Phys. Rev. A **54**, 1853 (1996).
[25] E.E. Salpeter, Phys. Rev. **87**, 328 (1952).
[26] B.E. Lautrup, A.Peterman, E.de Rafael, Phys. Rep. **3**, 193 (1972).
[27] S.A.Zapryagaev, N.L.Manakov, V.G.Pal'chikov, Theory of multi-charge ions with one and two electrons, Moscow, Energoatomizdat, 1985.
[28] H.F. Hameka, J. Chem. Phys. **47**, 2728 (1967).
[29] A.P. Martynenko, Phys. Rev. **A76**, 012505 (2007).
[30] A.P. Martynenko, Phys. Atom. Nucl. **71**, 125 (2008).
[31] M.I.Eides, H.Grotch, V.A.Shelyuto, Phys. Rev. D **65**, 013003 (2001).
[32] H.Bethe, E.Salpeter, Quantum mechanics of one- and two-electron atoms, Berlin, Springer-Verlag, 1957.
[33] G.Breit, Phys. Rev. **35**, 1447 (1930).
[34] D.C.Morton, Q.Wu and G.W.F.Drake, Physical Review A**73**, 034502 (2006).
[35] I.I.Sobel'man, Introduction to the theory of atomic spectra, Moscow, Fizmatlit, 1963.